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Modified singular manifold expansion: application to the Boussinesq and Mikhailov–Shabat systems

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Abstract. In this paper we present a unified treatment of a modified singular manifold expansion method as an improved variant of the Painlevé analysis for partial differential equations with two branches in the Painlevé expansion. We illustrate the method by fully applying it to the Boussinesq classical system and the Mikhailov–Shabat system.

1. Introduction

Integrability still seems to be a word with elusive meaning. Many efforts have been dedicated, in the last decade to the quest for a more definite and precise concept of integrability. It would be impossible (and outside the scope of this paper) any attempt to articulate a consistent framework encompassing all these efforts. We shall try to make some progress on one particular interconnection of this network: the relationship between integrability and the Painlevé property [1]. This is also closely related to the already classic paper by Weiss, Tabor and Carnevale [2] on the Painlevé test for partial differential equations.

Our first observation is based on the fact that Painlevé test for PDE has paved the way for establishing (as it has been shown in a variety of examples [3, 4]) the relation among the singular manifold method [5], based on the truncation of the Painlevé series, and Hirota's bilinear formalism based on the definition and use of the τ-function [6, 20, 21] which has been proved to be extremely successful for the explicit construction of N-soliton solutions.

One important drawback of this parallelism appears when this last approach uses, for the N-soliton construction, more than one τ-function. In this case the relationship among both approaches loses its meaning.

In this paper we give strong support to the conjecture that the number of τ-functions used in Hirota's method equals the number of branch expansions in Painlevé's singular manifold approach. For this to be true we shall need to generalize the latter to more

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than one singular manifold expansion. We shall do this in two cases: The classical Boussinesq system (CB) [7];

$$u_t + \omega_x + uu_x = 0 \tag{1.1a}$$

$$\omega_t + u_{xxx} + (u\omega)_x = 0 \tag{1.1b}$$

and the Mikhailov–Shabat system (MS) that appears in the classification of integrable systems given by these authors [8, 9];

$$p_t = p_{xx} + (p+q)q_x - (p+q)^3/6 \tag{1.2a}$$

$$-q_t = q_{xx} - (p+q)p_x - (p+q)^3/6. \tag{1.2b}$$

Hirota and Satsuma have constructed N-soliton solutions of both systems using the bilinear formalism [10]. On the other hand Sachs has analysed the Painlevé property for CB [11] and Flaschka, Newell and Tabor for MS [12]. Also Conte [13] has studied the Painlevé property and found a Lax pair for a version of MS in PDE form that will be described below.

In section 2 we shall be briefly reviewing the Painlevé test in the WTC version [2] for the CB system. The problems arising in the traditional singular manifold method will also be discussed. The rest of the section will be entirely dedicated to our conjecture and constitutes the core of the paper. We shall be introducing a new expansion using two singular manifolds and the two different expansion branches will be presented and discussed. Using our new procedure we shall be able to construct both the auto-Bäcklund transformations and its Lax pair for the CB system in section 3. As a bonus a method for generating solutions in an iterative manner using only linear equations will be briefly described. The relationship among Hirota's method and ours will also be presented. All this analysis will be applied again to the MS system in section 4 to provide further support to our conjecture. We close with some comments and prospects for further research along these lines.

2. The Boussinesq system and the singular manifold method

The classical Boussinesq system (CB):

$$u_t + \omega_x + uu_x = 0 \tag{2.1a}$$

$$\omega_t + u_{xxx} + (u\omega)_x = 0 \tag{2.1b}$$

is known to be an appropriate model for describing the behaviour of water waves in shallow channels [7]. On the theoretical side it is also known to be defined as the compatibility condition of a Lax pair [15] and to be solvable through the inverse scattering method [14, 15]. On the other hand Hirota has shown [16] that the bilinear formulation of (2.1) is a reduction of the modified KP equation. This allows us to obtain multisolitonic solutions for CB through the reduction of those obtained by Jimbo and Miwa [17] for the KP hierarchy.

On the other bank of the river Sachs has shown [11] that (2.1) has the Painlevé property. We shall briefly sketch here his main results. The Painlevé test in the WTC version [2] requires the functions u and ω be expanded in the form:

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t)\phi^{j-r}(x,t) \tag{2.2a}$$

$$\omega(x,t) = \sum_{j=0}^{\infty} \omega_j(x,t)\phi^{j-s}(x,t) \tag{2.2b}$$

where $\phi(x, t)$ is an arbitrary function: the celebrated movable singularity manifold. The exponents r and s and the functions u_0 and ω_0 in (2.2) are determined through the usual balance of dominant terms. If there exist several solutions of these balance equations one usually speaks of different expansion branches. For instance in the case of the system (2.1) one has the following solutions of the balance equations:

$$r=1 \tag{2.3a}$$

$$s=2 \tag{2.3b}$$

$$u_0=2a\phi_x \tag{2.3c}$$

$$\omega_0=-2\phi_x^2 \tag{2.3d}$$

but a can take the values ± 1 so that (2.1) has *two different expansion branches.* Sachs has shown [11] that (2.1) possesses the Painlevé property in both expansion branches. Also he has applied to this system the singular manifold method [5], i.e. assuming a truncated expansion we try to identify whether there exists a $\phi(x, t)$ verifying

$$u(x, t)=\sum_{j=0}^{r} u_j(x, t)\phi^{j-r}(x, t) \tag{2.4a}$$

$$\omega(x, t)=\sum_{j=0}^{s} \omega_j(x, t)\phi^{j-s}(x, t). \tag{2.4b}$$

The usual way to proceed after inserting (2.4) into (2.1) is to assume that *all coefficients for each power of ϕ vanish independently.* In so doing one obtains:

$$u_0=2a\phi_x \tag{2.5a}$$

$$u_1=-[\phi_t+a\phi_{xx}]/\phi_x \tag{2.5b}$$

$$\omega_0=-2(\phi_x)^2 \tag{2.5c}$$

$$\omega_1=2\phi_{xx} \tag{2.5d}$$

$$\omega_2=a(u_1)_x \tag{2.5e}$$

where $a=\pm 1$ and u_1 and ω_2 must be solutions of (2.1). This is why the truncated expansion (2.4) is actually an auto-Bäcklund transformation among solutions of (2.1).

However, as it was already pointed out by Sachs [11] the equations (2.5) restrict us to solutions of (2.1) that *additionally* verify the constraint $\omega=au_x$ in which case the system reduces to the Burgers equation:

$$u_t+uu_x+au_{xx}=0. \tag{2.6}$$

As it has been described, the singular manifold method is obviously inadequate to deal with several expansion branches and consequently for a possible interconnection with the Hirota bilinear method. We now propose an important modification of the method based upon the following considerations:

(1) Several criticisms have been already appeared in the literature concerning the singular manifold method as it has been described ([3, 12]). Specifically the assumption that the coefficients have to vanish independently for each power of ϕ has been seen as too much of a restrictive condition since all what we need to demand is that the truncation ansatz (2.4) must be satisfied. However this requirement could be fulfilled *just by setting to zero the sum of all contributions for each different power of ϕ.* Such modifications have already been discussed in the framework of some particular evolution equations ([4, 12]).

(2) On the other hand the connection between the singular manifold method and the Hirota formalism [4, 21] lies on the interpretation of (2.4) as an auto-Bäcklund transformation of the form

$$u(x,t)=\sum_{j=0}^{r-1} u_j(x,t)\phi^{j-r}(x,t)+u_r \tag{2.7a}$$

$$\omega(x,t)=\sum_{j=0}^{s-1} \omega_j(x,t)\phi^{j-s}(x,t)+\omega_s \tag{2.7b}$$

where (u_r, ω_s) and (u, ω) are solutions of the system (2.1). As any Bäcklund transformation this is just an iterative procedure for finding more complicated solutions starting with the simplest ones. The question is that having several expansion branches we make an unnecessary choice by using, as a seed solution, one of a definite branch. In doing so we will always stick to that branch henceforth neglecting all other solutions.

From all these considerations we propose to modify the singular manifold expansion in the following way. First we identify the number of independent expansion branches. After that we modify the expansion (2.4) by using as many singular manifolds as expansion branches. The generalized form of (2.4) for *two* expansion branches takes the form

$$u=\sum_{j=0}^{r-1} u_j\phi^{j-r}+\sum_{j=0}^{r-1} u_j'\sigma^{j-r}+u_r \tag{2.8a}$$

$$\omega=\sum_{j=0}^{s-1} \omega_j\phi^{j-s}+\sum_{j=0}^{s-1} \omega_j'\sigma^{j-s}+\omega_s \tag{2.8b}$$

where we have taken (u_0, ω_0) for one expansion branch and (u_0', ω_0') for the other.

If we make any attempt to express the solutions in the form of an expansion, as (2.8), we have necessarily to deal with crossed terms coming from ϕ and σ coefficients when inserting (2.8) into (2.1). We now remind the reader of the remarks we have made previously with regard to setting the coefficients to zero. Also, we have to find a way to rewrite auto-Bäcklund transformations of the kind above described if we want to be successful in relating Painlevé analysis and Hirota formalism. All these ideas will be developed to a large extent when specifically applied to the CB system. We will do this in the next section.

3. The method of the two singular manifolds for the Boussinesq system

Let us construct an auto-Bäcklund transformation of the form (2.8) for the CB system (2.1). Restricting ourselves for the moment to the $a=1$ branch we look for a truncated expansion of the form

$$u=2(\phi_x/\phi)+u_1 \tag{3.1a}$$

$$\omega=-2(\phi_x/\phi)^2+2(\phi_{xx}/\phi)+\omega_2 \tag{3.1b}$$

which after substitution in (2.1) yields

$$(u_1)_t+u_1(u_1)_x+(\omega_2)_x+2\{(\phi_{xx}/\phi)+(\phi_t/\phi)+u_1(\phi_x/\phi)\}_x=0 \tag{3.2a}$$

$$(\omega_2)_t+(u_1)_{xxx}+(\omega_2u_1)_x+2\{(\phi_x/\phi)_t+(\phi_{xx}/\phi)_x+\omega_2(\phi_x/\phi)+u_1(\phi_x/\phi)_x\}_x=0 \tag{3.2b}$$

Dropping for a moment the requirement that different powers of ϕ must have independent vanishing coefficients then (u_1, ω_2) do not have to be a set of solutions of CB and we can reinterpret (3.2) as a new system for which we again look for a truncated expansion. Since the dominant terms of (3.2) are of the same nature as those of (2.1) we now take the other branch $a=-1$ in the form:

$$u_1=-2(\sigma_x/\sigma)+\alpha \tag{3.3a}$$

$$\omega_2=-2(\sigma_x/\sigma)^2+2(\sigma_{xx}/\sigma)+\beta. \tag{3.3b}$$

Combining this last expression with (3.1) we easily find:

$$u=2\{(\phi_x/\phi)-(\sigma_x/\sigma)\}+\alpha \tag{3.4a}$$

$$\omega=2\{(\phi_x/\phi)+(\sigma_x/\sigma)\}_x+\beta. \tag{3.4b}$$

We now demand (α,β) that be a solution of (2.1). Automatically (3.4) will be the auto-Bäcklund transformation we are looking for. Substituting (3.3) in (3.2) we obtain for ϕ and σ the expressions

$$(\phi_x/\phi)[\alpha+v_1+w_1]-(\sigma_x/\sigma)[\alpha-v_2+w_2]=2(\sigma_x/\sigma)(\phi_x/\phi) \tag{3.5a}$$

$$(\phi_x/\phi)[\beta+v_{1x}+w_{1x}+\{v_1-(\phi_x/\phi)\}\{\alpha+v_1+w_1-2(\sigma_x/\sigma)\}]$$
$$-(\sigma_x/\sigma)[\beta+v_{2x}-w_{2x}-\{v_2-(\sigma_x/\sigma)\}\{\alpha-v_2+w_2+2(\phi_x/\phi)\}]=0 \tag{3.5b}$$

where v_i and w_i are defined as usual ([18, 19]):

$$v_1=(\phi_{xx}/\phi_x) \qquad w_1=(\phi_t/\phi_x) \tag{3.6a}$$

$$v_2=(\sigma_{xx}/\sigma_x) \qquad w_2=(\sigma_t/\sigma_x). \tag{3.6b}$$

We finally see through this procedure that (3.4) is an auto-Bäcklund transformation that allows us to generate a solution (u,ω) starting from another known solution (α,β) within the framework of the two singular manifolds ϕ and σ verifying (3.5).

Furthermore, the pair of equations (3.5) can greatly be simplified by substituting (3.5*a*) in (3.5*b*) and also in the derivative of the former. After this step we impose the coefficients of ϕ_x and σ_x to vanish. Then we obtain:

$$\alpha=-v_1-w_1+v_2-w_2 \tag{3.7a}$$

$$\beta=\tfrac{1}{2}[w_1^2+w_2^2-v_1^2-v_2^2] \tag{3.7b}$$

$$v_{1x}+w_{1x}+\tfrac{1}{2}(v_1+w_1)(v_2-w_2-v_1+w_1)=0 \tag{3.7c}$$

$$v_{2x}-w_{2x}+\tfrac{1}{2}(v_2-w_2)(-v_2-w_2+v_1+w_1)=0. \tag{3.7d}$$

We recall that (α,β) must be solutions of the CB system. Supposing (3.7*a*, *b*) satisfy (2.1) we are lead to the following relation:

$$(v_1+w_1+v_2-w_2)_t=[w_1(v_1+w_1)+w_2(v_2-w_2)]_x. \tag{3.8}$$

A crucial observation at this point is that (3.8) can be written as the compatibility condition of a linear system in the form:

$$\Psi_x=b_0[v_1+w_1+v_2-w_2]\Psi \tag{3.9a}$$

$$\Psi_t=b_0[w_1(v_1+w_1)+w_2(v_2-w_2)]\Psi. \tag{3.9b}$$

Taking $b_0=\frac{1}{4}$ we are able to express (3.9) as a function of the solutions (α,β) in the form:

$$\Psi_{xx}=[(\alpha^2/16)-(\beta/4)]\Psi \tag{3.10a}$$

$$\Psi_t=(\alpha_x/4)\Psi-(\alpha/2)\Psi_x. \tag{3.10b}$$

Using (3.7a) and (3.9a) and taking into account (3.6) we obtain, for ϕ and σ, the following *linear equations*:

$$\phi_{xx}+\phi_t+[(\alpha/2)-2(\Psi_x/\Psi)]\phi_x=0 \tag{3.11a}$$

$$\sigma_{xx}-\sigma_t-[(\alpha/2)+2(\Psi_x/\Psi)]\sigma_x=0. \tag{3.11b}$$

Let us briefly summarize the procedure we have found to generate solutions of the CB system:

(1) We begin with a known solution (α, β) of (2.1). Inserting this in (3.10) we obtain $\Psi(x, t)$.

(2) With $\Psi(x, t)$ we solve the *linear equations* (3.11) obtaining $\phi(x, t)$ and $\sigma(x, t)$. However, we should recall that ϕ and σ must also satisfy (3.5a) since, so far, only the derivative of this equation had been used.

(3) Finally we can generate a new solution (u, ω) using the auto-Bäcklund transformation (3.4).

Let us apply this procedure to the CB system (2.1) using the following trivial solution

$$\alpha=\alpha_0=\text{constant} \tag{3.12a}$$

$$\beta=\beta_0=\text{constant.} \tag{3.12b}$$

A solution of (3.10) in this case appears as:

$$\Psi(x, t)=\exp\{k(x-vt)\} \tag{3.13}$$

where

$$v=(\alpha_0/2) \tag{3.14a}$$

$$\beta_0=v^2-4k^2. \tag{3.14b}$$

Now the solutions of (3.11) are:

$$\phi(x, t)=A_0+\sum A_n \exp\{2k_n(x-v_nt)\} \tag{3.15a}$$

$$\sigma(x, t)=B_0+\sum B_n \exp\{2k'_n(x-v'_nt)\} \tag{3.15b}$$

where A_0, B_0, A_n, B_n, are arbitrary constants while v_n, k_n, v'_n and k'_n satisfy:

$$2k_n-v_n=2k-v \tag{3.16a}$$

$$2k'_n+v'_n=2k+v. \tag{3.16b}$$

But ϕ and σ must also satisfy (3.5a). Inserting (3.15) in (3.5a) we obtain a different set of solutions depending on whether $\beta_0=0$ or $\beta_0\neq 0$. We shall be dealing with these two cases separately:

(I) $\beta_0=0$. According to (3.14b) $\beta_0=0$ implies $v=\pm 2k$. We have to distinguish also among these two cases:

(I.a) $v=-2k$. In this case (3.5a) leads to the following solutions for ϕ and σ:

$$\phi(x, t)=A\exp\{2k(x+2kt)\} \tag{3.17a}$$

$$\sigma(x, t)=B_0+\sum B_n \exp\{2k'_n(x+2k'_nt)\} \tag{3.17b}$$

and then the solution (3.4) takes the form:

$$u(x, t)=-2(\sigma_x/\sigma) \tag{3.18a}$$

$$\omega(x, t)=-u_x. \tag{3.18b}$$

Using (3.18*b*) in (2.1) we can easily reduce the CB system to the Burgers equation. Also (3.18*a*) corresponds to a particular solution of this equation describing the confluence of shock waves [7].

(I.b) $v=+2k$. Using the same procedure as before we obtain:

$$\Phi(x, t)=A_0+\sum A_n \exp\{2k_n(x-2k_n t)\} \tag{3.19a}$$

$$\sigma(x, t)=B \exp\{2k(x-2kt)\} \tag{3.19b}$$

$$u(x, t)=2(\phi_x/\phi) \tag{3.19c}$$

$$\omega(x, t)=u_x \tag{3.19d}$$

that corresponds through (3.19*d*) to another reduction to Burgers equation.

(II) $\beta_0\neq 0$. In this case equation (3.5*a*) leads to

$$k_n=k'_n=k \tag{3.20a}$$

$$v_n=v'_n=v \tag{3.20b}$$

$$(\phi_x/\phi)(v+2k)-(\sigma_x/\sigma)(v-2k)=2(\phi_x/\phi)(\sigma_x/\sigma) \tag{3.20c}$$

so that we obtain

$$\phi(x, t)=A_0[1+q \exp\{2k(x-vt)+\varphi_0\}] \tag{3.21a}$$

$$\sigma(x, t)=B_0[1+p \exp\{2k(x-vt)+\varphi_0\}] \tag{3.21b}$$

where p and q are given by

$$q=v+2k \tag{3.22a}$$

$$p=v-2k \tag{3.22b}$$

and finally the solution can be written as

$$u=2v+2[(\phi_x/\phi)-(\sigma_x/\sigma)] \tag{3.23a}$$

$$\omega=v^2-4k^2+2[(\phi_x/\phi)+(\sigma_x/\sigma)]_x. \tag{3.23b}$$

This is the soliton founded by Kaup [15] using the inverse scattering method and by Hirota [16] through the bilinear formalism.

At this point one can easily establish the relationship with Hirota bilinear formalism. From the initial solution (α_0, β_0) we apply the transformation (3.4) an arbitrary number of times following the procedure already described in each one of the steps. After n steps we obtain

$$\begin{aligned} u^{(n)}&=2\{[(\phi_n)_x/\phi_n]-[(\sigma_n)_x/\sigma_n]\}+u^{(n-1)}\\ &=2\{[(\phi_n)_x/\phi_n]-[(\sigma_n)_x/\sigma_n]\}+\dots 2\{[(\phi_1)_x/\phi_1]-[(\sigma_1)_x/\sigma_1]\}+\alpha_0 \end{aligned} \tag{3.24a}$$

$$\begin{aligned} \omega^{(n)}&=2\{[(\phi_n)_x/\phi_n]+[(\sigma_n)_x/\sigma_n]\}_x+\omega^{(n-1)}\\ &=2\{[(\phi_n)_x/\phi_n]+[(\sigma_n)_x/\sigma_n]\}_x+\dots 2\{[(\phi_1)_x/\phi_1]+[(\sigma_1)_x/\sigma_1]\})_x+\beta_0. \end{aligned} \tag{3.24b}$$

To find both sets of τ-functions we just define:

$$\tau = \phi_1 \phi_2 \ldots \phi_n \tag{3.25a}$$

$$\tau' = \sigma_1 \sigma_2 \ldots \sigma_n \tag{3.25b}$$

and the solutions of the CB system can be now expressed in the form

$$u = \alpha_0 + 2[\log(\tau/\tau')]_x \tag{3.26a}$$

$$\omega = \beta_0 + 2[\log(\tau\tau')]_{xx} \tag{3.26b}$$

as required by the Hirota bilinear formalism [16].

A final observation concerns the Galilean invariance of the CB system. Under the transformation:

$$x' = x - \lambda t \tag{3.27a}$$

$$u' = u - \lambda \tag{3.27b}$$

the equations (2.1) remain invariant. This symmetry can be used to introduce the spectral parameter in the linear system (3.10). To see this one can just apply the transformation (3.27) to (3.10) obtaining:

$$\Psi_{xx} = [((\alpha+\lambda)/4)^2 - (\beta/4)]\Psi \tag{3.28a}$$

$$\Psi_t = (\alpha_x/4)\Psi - ((\alpha-\lambda)/2)\Psi_x. \tag{3.28b}$$

This is exactly the Lax pair found by Jaulent and Miodek [14].

4. The method of the two singular manifolds for the Mikhailov–Shabat system

As already noted in section 1, the system given by the equation (1.2) was first classified as an integrable system by Mikhailov and Shabat ([8, 9]). We shall be referring to it as the MS system. Let us introduce the transformation

$$u = p + q \tag{4.1a}$$

$$\omega = q_x - p_x. \tag{4.1b}$$

Then, the MS system (1.2) becomes

$$u_t + \omega_x - uu_x = 0 \tag{4.2a}$$

$$\omega_t + u_{xxx} + u\omega_x + u_x\omega - u^2 u_x = 0. \tag{4.2b}$$

Now we set $u = v_x$. After substitution, (4.2) transform into the single PDE:

$$v_{tt} - v_{xxxx} + v_t v_{xx} - \tfrac{1}{2} v_x^2 v_{xx} = 0. \tag{4.3}$$

This equation also corresponds to one of the modified Boussinesq equations found by Hirota and Satsuma [10]. The singular manifold analysis has been carried out by Conte [13]. Now we shall apply to this equation our method by first taking into account the number of expansion branches. A simple look at the dominant terms of the expansion (2) for (4.3) leads to the following choices:

$$r = 1 \tag{4.4a}$$

$$s = 2 \tag{4.4b}$$

$$u_0 = a\gamma\phi_x \tag{4.4c}$$

$$\omega_0 = -6\phi_x^2 \tag{4.4d}$$

where $\gamma^2 = -12$ and again a can be +1 or −1. We again face a two expansion branch problem. As in the previous case we shall assume the existence of two different singular manifolds ϕ and σ that allow a truncation solution of (4.2) in the form:

$$u = \gamma[(\phi_x/\phi) - (\sigma_x/\sigma)] + \alpha \tag{4.5a}$$

$$\omega = 6[(\phi_x/\phi) + (\sigma_x/\sigma)]_x + \beta \tag{4.5b}$$

where (α, β) is a set of solutions of the same MS system (4.2). Inserting (4.5) into (4.2) we obtain for ϕ and σ the following equations:

$$(\phi_x/\phi)[\gamma w_1 + 6v_1 - \gamma\alpha] + (\sigma_x/\sigma)[-\gamma w_2 + 6v_2 + \gamma\alpha] = 12(\sigma_x/\sigma)(\phi_x/\phi) \tag{4.6a}$$

$$\begin{aligned}(\phi_x/\phi)\{6(w_{1x} + w_1 v_1) + \gamma(v_{1x} + v_1^2) + \gamma\beta + 6v_1\alpha - \gamma\alpha^2\} \\ + (\sigma_x/\sigma)\{6(w_{2x} + w_2 v_2) - \gamma(v_{2x} + v_2^2) - \gamma\beta + 6v_2\alpha + \gamma\alpha^2\} \\ + (\phi_x^2/\phi^2)\{3\gamma v_1 + 6\alpha - 6w_1\} + (\sigma_x^2/\sigma^2)\{-3\gamma v_2 + 6\alpha - 6w_2\} \\ + (\phi_x/\phi)(\sigma_x/\sigma)\{6\gamma v_2 - 6\gamma v_1 + 6\gamma(\sigma_x/\sigma) - 6\gamma(\phi_x/\phi) - 24\alpha\} = 0.\end{aligned} \tag{4.6b}$$

As in section 3 v_i and w_i are defined by (3.6). We now combined the previous equations in the following form: substitute (4.6*a*) into (4.6*b*) and also in the derivative of (4.6*a*). After that we set to zero all coefficients in σ_x and ϕ_x. A rather tedious and cumbersome calculation yields:

$$\alpha = -w_1 - w_2 - (\gamma/2)v_1 + (\gamma/2)v_2 \tag{4.7a}$$

$$2\beta = -4v_{1x} - 4v_{2x} - v_1^2 + w_1^2 - v_2^2 + w_2^2 \tag{4.7b}$$

$$\gamma w_{1x} + \gamma w_{2x} - 2v_{1x} + 2v_{2x} + v_1^2 - v_2^2 - w_1^2 + w_2^2 = 0 \tag{4.7c}$$

$$\begin{aligned}\gamma w_{1x} - \gamma w_{2x} + 6v_{1x} + 6v_{2x} - 3(v_1 - v_2)^2 - (w_1 - w_2)^2 \\ + \gamma(w_1 v_1 - w_2 v_2) + \gamma(w_1 + w_2)(v_1 - v_2) = 0.\end{aligned} \tag{4.7d}$$

Since (α, β) must also be a solution of (4.2) we suppose that equation (4.2) must be satisfied by (4.7). Then we arrive at the condition:

$$\begin{aligned}[w_1 - w_2 + (\gamma/2)v_1 + (\gamma/2)v_2]_t \\ = [2v_{1x} - 2v_{2x} + (v_1/2)(\gamma w_1 - 2v_1) + (v_2/2)(\gamma w_2 + 2v_2)]_x.\end{aligned} \tag{4.8}$$

Again, as in section 3, (4.8) is nothing but the compatibility condition of the linear system:

$$\Psi_x = b_0[w_1 - w_2 + (\gamma/2)v_1 + (\gamma/2)v_2]\Psi \tag{4.9a}$$

$$\Psi_t = b_0[2v_{1x} - 2v_{2x} + (v_1/2)(\gamma w_1 - 2v_1) + (v_2/2)(\gamma w_2 + 2v_2)]\Psi. \tag{4.9b}$$

Setting $b_0 = (2\gamma)^{-1}$ and with the help of (4.7) one can easily re-express (4.9) just in terms of (α, β) in the form:

$$\Psi_{xx} = [(\alpha^2/16) - (\beta/4)]\Psi \tag{4.10a}$$

$$\Psi_t = (\alpha_x/4)\Psi - (\alpha/2)\Psi_x. \tag{4.10b}$$

We call the reader's attention to the curious fact that (4.10) represents exactly the same Lax pair as that for the CB system [see (3.10)]. There is only one, quite important, difference. The spectral parameter has been dropped from (4.10). As we have already discussed one can use Galilean invariance to reinstate the spectral parameter in the CB system. Nevertheless the MS system lacks such an invariance.

The question of the spectral parameter is certainly an intriguing one but for us it is now meaningless since we can generate the solutions from (4.10) just as well. Starting from some known solution (α, β) we solve (4.10). Then using (4.7a) and (4.9a) we find ϕ and σ from the pair of *linear* equations

$$\gamma\phi_t - 6\phi_{xx} + [(\gamma\alpha/2) + 12(\Psi_x/\Psi)]\phi_x = 0 \tag{4.11a}$$

$$\gamma\sigma_t + 6\sigma_{xx} + [(\gamma\alpha/2) - 12(\Psi_X/\Psi)]\sigma_x = 0. \tag{4.11b}$$

We also have to check whether ϕ and σ satisfy (4.6a). Finally, the auto-Bäcklund transformation (4.5) provides us a new solution (u, ω).

As in the previous section we choose $(\alpha, \beta) = (\alpha_0, \beta_0) = $ (constants). Then, we obtain from (4.10)

$$\Psi(x, t) = \exp\{k(x - vt)\} \tag{4.12a}$$

with

$$v = (\alpha_0/2) \tag{4.12b}$$

$$\beta_0 = v^2 - 4k^2 \tag{4.12c}$$

while the solution of (4.11) [together with (4.6a)] yields for ϕ and σ the expressions

$$\phi(x, t) = 1 + q \exp\{2k(x - vt + \varphi_0)\} \tag{4.13a}$$

$$\sigma(x, t) = 1 + \rho \exp\{2k(x - vt + \varphi_0)\} \tag{4.13b}$$

where φ_0 is an arbitrary constant. Also q and p are given by

$$q = \gamma v + 4k \tag{4.14a}$$

$$p = \gamma v - 4k \tag{4.14b}$$

and finally from (4.5) we obtain the (u, ω) solution in the form

$$u = 2v + (8k^2 A_0/v)\{A_0 + \coth[2k(x - vt + \theta_0)]\}^{-1} \tag{4.15a}$$

$$w = v^2 - 4k^2 + 24k^2\{1 + A_0 \coth[2k(x - vt + \theta_0)]\}\,\{A_0 + \coth[2k(x - vt + \theta_0)]\}^{-2} \tag{4.15b}$$

where

$$A_0 = [3v^2/(3v^2 + 4k^2)] \tag{4.16a}$$

and

$$\exp(2k\theta_0) = (pq)^{1/2} \exp(2k\varphi_0). \tag{4.16b}$$

Notice that in spite of γ being an imaginary constant, the obtained solitonic solution is indeed real.

5. Conclusions

In this paper we have shown through two detailed examples that the conjecture of the *multiple singular manifold expansion* for integrable systems with *multiple expansion banches* definitely works. The more general question as to whether this can be generally proved is of course much more delicate but we believe we have presented a handful of good reasons to convince the reader that this general proof can be given shortly.

We should not forget however other less fundamental but equally important problems we have left partially open. From our point of view the most interesting one is the question of symmetries. At first sight one could be tempted to extrapolate the fact that the two Lax pairs are the same as to speculate that both dynamics are equivalent. The set of different symmetries constitutes a serious drawback to this assumption. Obviously this set of symmetries is connected with how one should introduce the spectral parameter and this point makes all the difference between the integrable system we are interested in.

Thus, it will be of primary importance to add to our consideration a thorough study of the symmetry group leaving the system invariant in order to see where and how the question of the invariance became so important as to arise as the main dynamical difference among apparently equivalent integrable systems. Work in this direction is also in progress.

Another interesting open question refers to the poly-Painlevé criterion introduced by Kruskal and Clarkson [1]. This method is used to examine simultaneously several branch points of an ODE. It will be very interesting in the near future to consider this method for stationary versions of the CB and MB systems.

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